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## Optimal guaranteed cost control of uncertain discrete time-delay systems

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### Abstract

This paper considers the problem of robust guaranteed cost control of linear discrete time-delay systems with parametric uncertainties. By matrix inequality approach, the robust quadratic stability of the system is studied. A control design method is developed such that the closed-loop system with a cost function has an upper bound irrespective of all admissible parameter uncertainties and unknown time delays. Furthermore, the upper bound (cost) can be optimized by incorporating with a minimization problem. A numerical example is given to show the potential of the proposed techniques.

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## 1. Introduction

Control systems design that can handle model uncertainties has been one of the most challenging problems, and has received considerable attention from control engineers and scientists in the past decades. There are two major issues in robust controller designs. The first is concerned with the robust stability of the uncertain closed-loop system (see, e.g., [11,2,31,30] and the references therein), and the other is the robust performance (see, e.g., Shi [26,27,29,32,33,37]). Note that the latter is more important since when controlling a system dependent on uncertain parameters, it is always desirable to design a control system which is not only stable, but also guarantees an adequate level of performance. Since the work of Chang and Peng [3], this issue has been addressed extensively, for example, to name a few, by Peres et al. [20,22] for continuous-time case, and Geromel et al. [7], Xie and Soh [39], Petersen et al. [24] for discrete-time case. On the other hand, the study of time-delay systems has received considerable interest during the past years (see, e.g., [35]). Time delay is commonly encountered in various engineering systems and is frequently a source of instability and poor performance [16]. In the work of Gutman and Palmor [8], nonlinear state feedback controllers have been considered, whereas Hasanul Basher et al. [9] has focused on memoryless linear state feedback. Recently, memoryless stabilization and  $H_\infty$  control of uncertain continuous-time-delay systems have been extensively investigated. For some representative prior work on this general topic, we refer the reader to [1,12–15,17,18,25]. The problem of robust stabilization for a class of time-varying delay systems with both Lipschitz and non-Lipschitz bounded uncertainties has been studied by Nguang [18] via Riccati equation approach, and a memoryless state feedback controller is designed. In the research conducted in [15], quadratic stabilization of continuous time systems with state-delay and norm-bounded time-varying uncertainties has been considered. More recently, the issue of delay-dependent robust stability and stabilization of uncertain linear delay systems has been tackled by Li and de Souza [14] via a linear matrix inequality approach. However, to the best of authors' knowledge, the problem of guaranteed cost control for linear discrete time-delay systems with parameter uncertainties has not been fully investigated yet. It should be noted that some attempt has been made in [17] to cope with the optimal quadratic guaranteed cost control of uncertain continuous time-delay systems, while the problem of sub-optimal guaranteed cost control for uncertain continuous time-delay systems has been studied by linear matrix inequality approach in [4]. Also, in the work by Kapila and Haddad [10], the problem of  $H_\infty$  stabilization of discrete-time system with state delays and exogenous bounded-energy  $\ell_2$  disturbances has been studied using Riccati equation approach, and sufficient conditions for the solution has been provided. In the meantime, Xu et al. [40] addressed the problems of quadratic stability and stabilization of uncertain linear discrete-time systems with state delay.

In this paper, motivated by the results of guaranteed cost control of discrete-time systems with norm bounded parameter uncertainties [24,39], and the counterpart of continuous time delay systems [17], we are mainly concerned with the problems of robust quadratic stability and robust quadratic guaranteed cost control for a class of linear discrete-time uncertain systems. The parameter uncertainty under consideration is real time-varying and norm-bounded. Inspired by the results without time-delays, we introduce the notion of *quadratic stability* for the time-delay system. A necessary and sufficient condition is given for the robust quadratic stability of uncertain systems for all admissible uncertainties. The performance adopted is the well-known quadratic cost. A state feedback control is designed such that the cost of the system is guaranteed to be within a certain bound for

all admissible uncertainties. Both finite and infinite horizon cases are considered. It is shown that the control problem can be solved if some matrix inequalities have positive definite solutions.

It should be remarked that the key difference between the work in [17] and present paper is that in [17] the results are on continuous-time systems, while this paper considers the counterpart of discrete-time systems. In particular, the techniques used here are different from those in [17], although both papers employ Riccati inequality approach. We believe the results obtained in this paper on discrete-time systems are of the same importance in both theory and practice as those in [17] on continuous-time systems.

This paper is organized as follows: in Section 2, the system under study is introduced and the problems we are going to solve are formulated. Also, some preliminary results are recalled. The problem of robust quadratic stability is tackled in Section 3, while matrix inequality type conditions are given. In addition, a result is provided to characterize all quadratic cost matrices. In Section 4, robust guaranteed cost controls are designed for both finite and infinite horizons, and it is concluded that upper bound (cost) of the cost function can be minimized by solving an optimization problem. A numerical example is included in Section 5 to demonstrate the usefulness of the developed theoretic results.

*Notation:* The notation used in this paper is quite standard.  $\mathcal{Z}$ ,  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  denote, respectively, the set of integer numbers, the  $n$  dimensional Euclidean space and the set of all  $n \times m$  real matrices. The superscript “T” denotes the transpose and the notation  $X \geq Y$  (respectively,  $X > Y$ ) where  $X$  and  $Y$  are symmetric matrices, means that  $X - Y$  is positive semi-definite (respectively, positive definite).  $I$  is the identity matrix with compatible dimension.  $E\{\cdot\}$  denotes the expectation operator with respect to some probability measure.  $P \operatorname{tr}(A)$  denotes the trace of a square matrix  $A$ .

## 2. Problem formulation and preliminaries

Consider the following class of dynamical system:

$$\begin{aligned} x_{k+1} &= (A + \Delta A_k)x_k + A_d x_{k-d} + (B + \Delta B_k)u_k, \\ x_s &= 0, \quad s < 0, \quad x_0 = x(0), \quad k \in \mathcal{Z}, \end{aligned} \quad (2.1)$$

where  $x_k \in \mathbb{R}^n$  is the state,  $u_k \in \mathbb{R}^m$  is the control input.  $A$ ,  $A_d$  and  $B$  are known constant matrices of appropriate dimensions,  $d > 0$  is an unknown scalar standing for time-delay, and  $\Delta A_k$  and  $\Delta B_k$  are unknown matrices which represent time-varying parametric uncertainties and assumed to belong to certain compact sets.

**Remark 2.1.** The motivation we consider system (2.1) containing uncertainties  $\Delta A_k$  and  $\Delta B_k$  stems from the fact that, in practical, it is almost impossible to get an exact mathematical model of a dynamical system due to the complexity of the system, the difficulty of measuring various parameters, environmental noises, uncertain and/or time-varying parameters, etc. Indeed, the model of the system to be controlled almost always contains some type uncertainty.

The admissible parameter uncertainties are assumed to be of the following forms:

$$[\Delta A_k \ \Delta B_k] = HF_k[E_1 \ E_2], \quad (2.2)$$

where  $H, E_1$  and  $E_2$  are known constant matrices of appropriate dimensions, and  $F_k$  is an unknown time-varying matrix satisfying the following bound condition:

$$F_k^T F_k \leq I \quad \forall k \geq 0. \quad (2.3)$$

For the notational simplicity, in the sequel, for a time-varying matrix  $X_k$ , we sometimes denote it by  $X$  wherever no confusion arises regarding the time dependence of this quantity.

**Remark 2.2.** The parameter uncertainty structure as in (2.2)–(2.3) has been widely used in the problems of robust control and robust filtering of uncertain systems, see, e.g., [11,19,23,26,29,28,33,38] and the references therein. It comprises the well-known “matching conditions” and many practical systems possess parameter uncertainties which can be either exactly modeled, or overbounded by (2.3). The uncertain matrix  $F_k$ ,  $k \in \mathcal{Z}$  containing the uncertain parameters in the state and input matrices, are allowed to be state dependent as long as (2.3) is satisfied along all possible state trajectories. The matrices  $H$ ,  $E_1$  and  $E_2$  specify how the uncertain parameter  $F_k$  affect the nominal matrix of system (2.1). Observe that the unit overbound for  $F_k$  does not cause any loss of generality. Indeed,  $F_k$  can be always normalized, in the sense of (2.3), by appropriately choosing the matrices  $H$ ,  $E_1$  and  $E_2$ . Also, it is worthwhile to mention that for technical and presentational simplicity, we consider only, in this paper, the uncertainties appearing in state matrix  $A$  and control input matrix  $B$ . Indeed, it is not difficult to add uncertainties, for instance,  $\Delta A_{dk}$ , in the delayed state matrix  $A_d$ , and obtain some similar results, which will take the known information from  $\Delta A_{dk}$  into account. Another fact is that since  $d$  is unknown, which makes the whole term “ $A_d x_{k-d}$ ” is kind of unknown already.

**Remark 2.3.** It should be noted that the increasing (widely) use of digital computers in control systems has led to considerable activity in the field of discrete-time and digital control systems. Hence the problems presented in this paper are meaningful in both theory and practice. Furthermore, system (2.1) encompasses many state space models of delay systems and can be used to represent many important physical systems; for example, cold rolling mills, wind tunnel and water resources systems, see, e.g., [16] and the references therein.

Motivated by the well-known linear quadratic control theory, we define the following cost function for uncertain system (2.1):

$$J(N, d, \Delta A_k, \Delta B_k) = x_N^T S x_N + \sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k), \quad (2.4)$$

where  $S > 0$ ,  $Q > 0$  and  $R > 0$  are given weighting matrices.

In the case of infinite horizon, i.e.,  $N \rightarrow \infty$ , the cost function of (2.4) will be replaced by

$$J(\infty, d, \Delta A_k, \Delta B_k) = \sum_{k=0}^{\infty} (x_k^T Q x_k + u_k^T R u_k). \quad (2.5)$$

In this paper, we study the problems of quadratic stability and quadratic guaranteed cost control associated with system (2.1) and cost function (2.4), (or (2.5)). By using matrix inequality approach and quadratic Lyapunov function, an upper bound on the cost function (2.4) (or (2.5)) is provided.

Before ending this section, let us recall the following Schur complements lemma.

**Lemma 2.1.** Given constant matrices  $M$ ,  $L$ ,  $Q$  of appropriate dimensions where  $M$  and  $Q$  are symmetric, then  $Q > 0$  and  $M + L^T Q^{-1} L < 0$  if and only if

$$\begin{bmatrix} M & L^T \\ L & -Q \end{bmatrix} < 0$$

or equivalently

$$\begin{bmatrix} -Q & L \\ L^T & M \end{bmatrix} < 0.$$

### 3. Analysis of robust performance

In this section, results on quadratically stable and quadratic cost for free system (2.1) ( $u_k \equiv 0$ ) with the cost function (2.5) are presented. First we introduce the following definition.

**Definition 3.1.** The uncertain delay system (2.1) (setting  $u_k \equiv 0$ ) is said to be quadratically stable if there exist symmetric positive definite matrices  $P$  and  $X$  such that

$$A_d^T P A_d - X < 0, \quad (3.1)$$

$$\begin{bmatrix} -P + X + \hat{A}^T P \hat{A} & \hat{A}^T P A_d \\ A_d^T P \hat{A} & A_d^T P A_d - X \end{bmatrix} < 0 \quad (3.2)$$

for all  $F_k$ ,  $F_k^T F_k \leq I$ , where  $\hat{A} = A + H F_k E_1$ .

**Remark 3.1.** It should be pointed out that Definition 3.1 extends the concept of quadratic stability to the case of uncertain linear time-delay system (2.1). Note that this definition is independent of the time delay  $d$ . Moreover, when the time delay term disappears, i.e.,  $A_d = 0$ , the above definition reduces to the standard definition of quadratic stability for uncertain linear systems with norm bounded uncertainty, see for example [6,36].

The following theorem gives the motivation for the definition of quadratic stability.

**Theorem 3.1.** Consider system (2.1) with  $u_k \equiv 0$  and the function

$$V_k(x_k, d) = x_k^T P x_k + \sum_{i=k-d}^{k-1} x_i^T X x_i. \quad (3.3)$$

If system (2.1) is quadratically stable, then there exists a Lyapunov function of the form (3.3) such that

$$\Delta V = V_{k+1} - V_k < 0,$$

hence  $x_k \rightarrow 0$  as  $k \rightarrow \infty$  (this is because the fact that system (2.1) is quadratically stable implies that  $\lim_{k \rightarrow \infty} x_k = 0$  and  $\lim_{k \rightarrow \infty} \Delta V = V(x_\infty) - V(x_0) = -V(x_0) < 0$ ).

Conversely, if there exist matrices  $P > 0$  and  $X > 0$  such that

$$\Delta V = V_{k+1} - V_k < 0,$$

then the system is quadratically stable.

**Proof.** First, since system (2.1) is quadratically stable, there exist  $P > 0$  and  $X > 0$  satisfying (3.1) and (3.2). Choose the Lyapunov functional candidate of the form (3.3) for system (2.1), and one has

$$\begin{aligned} \Delta V &= V_{k+1} - V_k \\ &= x_k^T \hat{A}^T P \hat{A} x_k + x_k^T \hat{A}^T P A_d x_{k-d} + x_{k-d}^T A_d^T P \hat{A} x_k \\ &\quad + x_{k-d}^T A_d^T P A_d x_{k-d} - x_k^T P x_k + x_k^T X x_k - x_{k-d}^T X x_{k-d} \\ &= x_k^T (-P + \hat{A}^T P \hat{A} + \hat{A}^T P A_d (X - A_d^T P A_d)^{-1} A_d^T P \hat{A} + X) x_k \\ &\quad - (x_k^T \hat{A}^T P A_d (X - A_d^T P A_d)^{-1} - x_{k-d}^T) (X - A_d^T P A_d) \\ &\quad \times (x_k^T \hat{A}^T P A_d (X - A_d^T P A_d)^{-1} - x_{k-d}^T)^T, \end{aligned} \quad (3.4)$$

where  $\hat{A} = A + H F_k E_1$ . By Lemma 2.1, the first term in the right-hand side of the equality (3.4) is less than zero for any  $x_k \neq 0$ , and the second term in the right-hand side of the equality (3.4) is always less than or equal to zero for any  $x_k$  and  $x_{k-d}$ . It follows that  $\Delta V < 0$  for all nonzero  $x_k$ .

Similarly, from (3.4), the second part of the theorem can be worked out.  $\square$

**Remark 3.2.** It can be seen that the quadratic stability problem is closely related to a Lyapunov function of the form (3.3). A similar Lyapunov function has been used in [34] to handle with  $H_\infty$  control problem. The continuous counterpart of (3.3) is employed in [17] to solve the same problem for continuous uncertain time-delay systems.

**Definition 3.2.** A symmetric positive matrix  $P$  is said to be a quadratic cost matrix for system (2.1) (with  $u_k \equiv 0$ ) and cost function (2.5) if there exists a matrix  $X > 0$  such that  $P$  satisfies

$$A_d^T P A_d - X > 0, \quad (3.5)$$

$$\begin{bmatrix} -P + X + Q + \hat{A}^T P \hat{A} & \hat{A}^T P A_d \\ A_d^T P \hat{A} & A_d^T P A_d - X \end{bmatrix} < 0 \quad (3.6)$$

for all  $F_k$ ,  $F_k^T F_k \leq I$ , where  $\hat{A} = A + H F_k E_1$ .

Based on Definition 3.2, we have the following theorem.

**Theorem 3.2.** Suppose  $P > 0$  is a quadratic cost matrix for the uncertain system (2.1) and the cost function (2.5) with  $u_k \equiv 0$ . Then the system is quadratically stable and the cost function satisfies the bound

$$J(\infty, d, \Delta A_k, 0) \leq x_0^T P x_0. \quad (3.7)$$

Conversely, if system (2.1) is quadratically stable, then there exists a quadratic cost matrix for this system and cost function (2.5) with  $u_k \equiv 0$ .

**Proof.** By the definition of quadratic cost matrix  $P$  for system (2.1) and the cost function (2.5),  $P$  satisfies (3.5)–(3.6) for all admissible parameter uncertainty  $F_k$ . Then the quadratic stability of system (2.1) follows from the fact of  $Q > 0$ .

Now, along the same line as the proof in Theorem 3.1, one obtains

$$\Delta V = V_{k+1} - V_k < -x_k^T Q x_k.$$

Summing both sides of the above inequality from 0 to  $\infty$ , we have

$$\sum_{k=0}^{\infty} x_k^T Q x_k \leq V(x_0) - V(x_{\infty}). \quad (3.8)$$

Since it has been shown that the system is quadratically stable, it follows that  $V(x_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Hence, the inequality (3.8) implies that

$$J(\infty, d, \Delta A_k, 0) \leq x_0^T P x_0,$$

and the proof of the first part of the theorem is completed.

For the proof of the second part of the theorem, since system (2.1) is quadratically stable, there exists a matrix  $P > 0$  satisfies (3.1) and (3.2) for all admissible uncertainty  $F_k$ . Hence, by using Lemma 2.1, there exists a fixed scalar  $\varepsilon > 0$ , which is independent of  $d$  and  $F_k$  such that

$$\begin{aligned} \varepsilon A_d^T P A_d - \varepsilon X &< 0 \\ \begin{bmatrix} -\varepsilon P + \varepsilon X + Q + \varepsilon \hat{A} P \hat{A} & \hat{A}^T P A_d \\ A_d^T P \hat{A} & \varepsilon (A_d^T P A_d - X) \end{bmatrix} &< 0. \end{aligned}$$

Therefore, if letting  $\hat{X} = \varepsilon X$ , the matrix  $\hat{P} = \varepsilon P$  is a quadratic cost matrix for the cost function (2.5).  $\square$

Next we establish a result which characterizes all quadratic cost matrices in terms of two matrix inequalities. To begin with, we recall the following two lemmas which will be used in the proof of our main result in this section.

**Lemma 3.1** (Petersen and Hollot [23]). Given any matrices  $D, F$  and  $E$  with appropriate dimensions and  $x, y$ , then

$$\max\{(x^T D F E y)^2 : F^T F \leq I\} = x^T D D^T x y^T E^T E y.$$

**Lemma 3.2** (Petersen [21]). Let  $X, Y$  and  $Z$  be given symmetric matrices with appropriate dimensions such that  $X \geq 0$ ,  $Y < 0$  and  $Z \geq 0$ . Furthermore, assume that

$$(\eta^T Y \eta)^2 - 4(\eta^T X \eta)(\eta^T Z \eta) > 0$$

for all nonzero  $\eta$ . Then, there exists a constant  $\lambda > 0$  such that

$$\lambda^2 X + \lambda Y + Z < 0.$$

**Theorem 3.3.** A matrix  $P > 0$  is a quadratic cost matrix for system (2.1) and cost function (2.5) with  $u_k \equiv 0$  if and only if there exist a matrix  $X > 0$  and a scalar  $\varepsilon > 0$  such that

$$A_d^T P A_d - X < 0 \quad (3.9)$$

$$\begin{bmatrix} -P + \frac{1}{\varepsilon} E_1^T E_1 + X + Q & A^T \\ A & A_d^T P A_d - X + \varepsilon H H^T \end{bmatrix} < 0. \quad (3.10)$$

**Proof** (Necessity). Suppose  $P$  is a quadratic cost matrix for system (2.1) and cost function (2.5) with  $u_k \equiv 0$ , then there exists a  $X > 0$  such that (3.5) and (3.6) are true. Define

$$Z = \begin{bmatrix} -P + X + Q & A^T \\ A & A_d^T P A_d - X \end{bmatrix}, \quad M = \begin{bmatrix} 0 & E_1^T F_k^T H^T \\ H F_k E_1 & 0 \end{bmatrix}. \quad (3.11)$$

Then for any  $x \neq 0$  one has from (3.6)

$$x^T Z x + x^T M x < 0,$$

that is,

$$x^T Z x < -\{\max x^T M x, F_k^T F_k \leq I\} \leq 0.$$

Letting  $x = [x_1^T \quad x_2^T]^T$ , then the above inequality implies

$$\begin{aligned} x^T Z x &< -\left\{ \max [x_1^T \quad x_2^T] \begin{bmatrix} 0 & E_1^T F_k^T H^T \\ H F_k E_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, F_k^T F_k \leq I \right\} (\leq 0) \\ &= -\{\max(x_1^T H F_k E_1 x_2 + x_2^T E_1^T F_k^T H^T x_1), F_k^T F_k \leq I\} \\ &= -\{\max 2x_1^T H F_k E_1 x_2, F_k^T F_k \leq I\} \leq 0. \end{aligned} \quad (3.12)$$

By using Lemma 3.1, one obtains from (3.12)

$$\begin{aligned} (x^T Z x)^2 &> 4\{\max\{x_1^T H F_k E_1 x_2 : F_k^T F_k \leq I\}\}^2 \\ &\geq 4x_1^T H H^T x_1 x_2^T E_1^T E_1 x_2 \\ &= 4x^T \begin{bmatrix} H H^T & 0 \\ 0 & 0 \end{bmatrix} x x^T \begin{bmatrix} 0 & 0 \\ 0 & E_1^T E_1 \end{bmatrix} x. \end{aligned} \quad (3.13)$$



By applying Lemma 3.2 to (3.13), it follows that there exists a scalar  $\varepsilon > 0$  such that

$$\frac{1}{\varepsilon^2} \begin{bmatrix} HH^T & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{\varepsilon} Z + \begin{bmatrix} 0 & 0 \\ 0 & E_1^T E_1 \end{bmatrix} < 0. \quad (3.14)$$

The desired result now can be worked out by substituting  $Z$  of (3.11) into (3.14).

(Sufficiency) By the inequality

$$0 \leq x^T \begin{bmatrix} \frac{E_1^T}{\sqrt{\varepsilon}} \\ -\sqrt{\varepsilon} H F_k \end{bmatrix} \begin{bmatrix} \frac{E_1}{\sqrt{\varepsilon}} & -\sqrt{\varepsilon} F_k^T H^T \end{bmatrix} x$$

it is trivial to show that

$$\begin{bmatrix} \frac{1}{\varepsilon} E_1^T E_1 & 0 \\ 0 & \varepsilon H H^T \end{bmatrix} \geq \begin{bmatrix} 0 & E_1^T F_k^T H^T \\ H F_k E_1 & 0 \end{bmatrix}.$$

By substituting the above inequality into (3.10), we have (3.6), which completes the proof.  $\square$

#### 4. Guaranteed cost control

In this section, we will provide guaranteed cost control results for system (2.1) with cost function (2.4) (for finite horizon), or (2.5) (for infinite horizon).

It is well known that when there are no parametric uncertainties ( $\Delta A_k \equiv 0$  and  $\Delta B_k \equiv 0$ ) and no time delay ( $A_d = 0$ ), the cost of (2.4) for  $u_k \equiv 0$  is given by

$$J(N) = x_0^T P_0 x_0,$$

where  $P_0 > 0$  is calculated by the recursive equation

$$A^T P_{k+1} A - P_k + Q = 0, \quad P_N = S.$$

We call, as in the previous section,  $P_0$  a cost matrix. The above observation leads to the following definition of the guaranteed cost control.

**Definition 4.1.** A control law  $u_k = K_k x_k$  is said to be a quadratic guaranteed cost control associated with a cost matrix  $P_0$  for system (2.1) and the cost function (2.4) if there exist a positive definite matrix sequence  $P_k$ ,  $k = 0, 1, \dots, N$  and a positive definite matrix sequence  $X_k$ ,  $k = 0, 1, 2, \dots, N$  such that

$$A_d^T P_k A_d - X_k < 0 \quad (4.1)$$

$$\begin{bmatrix} -P_k + K_k^T R K_k + X_k + Q & [A + H F_k E_1 + (B + H F_k E_2) K_k]^T \\ [A + H F_k E_1 + (B + H F_k E_2) K_k] & A_d^T P_k A_d - X_k \end{bmatrix} < 0 \quad (4.2)$$

for all  $F_k$ ,  $F_k^T F_k \leq I$ .

In infinite horizon case, the definition of guaranteed cost control is given below.

**Definition 4.2.** A control law  $u = Kx_k$  is said to be a quadratic guaranteed cost control associated with a cost matrix  $P$  for system (2.1) and the cost function (2.5) if there exists a matrix  $X > 0$  such that

$$A_d^T P A_d - X < 0 \quad (4.3)$$

$$\begin{bmatrix} -P_k + K^T R K + X + Q & [A + H F_k E_1 + (B + H F_k E_2) K]^T \\ [A + H F_k E_1 + (B + H F_k E_2) K] & A_d^T P A_d - X \end{bmatrix} < 0 \quad (4.4)$$

for all  $F_k$ ,  $F_k^T F_k \leq I$ .

**Remark 4.1.** The above definitions are extensions of the concepts of guaranteed cost control for discrete-time systems without time delays [39]. It can be observed that the notion of quadratic guaranteed cost control is similar to the notion of quadratic stabilization, which has been widely used, although it is conservative in the sense of fixing  $P_k$  (or  $P$ ), to cope with time-varying uncertainties, see for example, [5,21] and the references therein. Furthermore, from Theorems 3.1 and 3.2, it can be easily seen that if  $u = Kx_k$  is a quadratic guaranteed cost control in the infinite horizon, then it is also a quadratically stabilizing control law. Conversely, a quadratically stabilizing control law will achieve a guaranteed cost.

The following theorem shows that a quadratic guaranteed cost control for system (2.1) will provide an upper bound on the cost function (2.4).

**Theorem 4.1.** Consider system (2.1) with the cost function (2.4) and suppose the control law  $u_k = K_k x_k$  is a quadratic guaranteed cost control associated with cost matrix  $P_0$ . Then, the closed-loop uncertain system

$$x_{k+1} = [A + \Delta A_k + (B + \Delta B_k) K_k] x_k + A_d x_{k-d} \quad (4.5)$$

achieves

$$J(N, d, \Delta A_k, \Delta B_k) < x_0^T P_0 x_0, \quad x_0 \neq 0 \quad (4.6)$$

for all admissible uncertainties  $\Delta A_k$  and  $\Delta B_k$ .

**Proof.** Since  $u_k = K_k x_k$  is a quadratic guaranteed cost control associated with cost matrix  $P_0$ , there exist a positive definite matrix sequence  $P_k$ ,  $k = 0, 1, \dots, N$  and a positive definite matrix sequence  $X_k$ ,  $k = 0, 1, 2, \dots, N$  satisfying (4.1) and (4.2). Let

$$V(x_k, d) = x_k^T P_k x_k + \sum_{i=k-d}^{k-1} x_i^T X_i x_i. \quad (4.7)$$

By similar techniques used in the proof of Theorems 3.1 and 3.2, one has

$$V(x_{k+1}) - V(x_k) < -(x_k^T Q x_k + u_k^T R u_k)$$

for nonzero  $x_k$ . Hence

$$\sum_{k=0}^{N-1} (x_k^T Q x_k + u_k^T R u_k) < V(x_0) - V(x_N) < x_0^T P_0 x_0$$

for all admissible  $\Delta A_k$  and  $\Delta B_k$ . The proof ends.  $\square$

In the infinite horizon case, we have the following result which can be carried out by a similar technique as that in Theorem 4.1.

**Theorem 4.2.** Consider system (2.1) with the cost function (2.5) and suppose the control law  $u = Kx_k$  is a quadratic guaranteed cost control associated with cost matrix  $P$ . Then, the closed-loop uncertain system

$$x_{k+1} = [A + \Delta A_k + (B + \Delta B_k)K]x_k + A_d x_{k-d}$$

achieves

$$J(\infty, d, \Delta A_k, \Delta B_k) < x_0^T P_0 x_0, \quad x_0 \neq 0$$

for all admissible uncertainties  $\Delta A_k$  and  $\Delta B_k$ .

Now, we propose a design method that could provide an optimal guaranteed cost for uncertain time-delay system (2.1) and the cost function (2.4), or (2.5). First, we deal with the finite horizon case.

**Theorem 4.3.** Consider system (2.1) with cost function (2.4). Suppose for some scaling sequence  $\varepsilon_k > 0$ ,  $k = 0, 1, \dots, N-1$ , there exist  $S_k > 0$ ,  $k = 0, 1, \dots, N-1$ ,  $S_N = S$  such that

$$A_d^T S_{k+1} A_d + \varepsilon_k H S_{k+1} H^T - I < 0 \quad (4.8)$$

$$\begin{bmatrix} -S_k + \frac{1}{\varepsilon_k} E_1^T E_1 + Q & A^T & A^T Y_{k+1} B + \frac{1}{\varepsilon_k} E_1^T E_2 \\ A & -Y_{k+1}^{-1} & 0 \\ (A^T Y_{k+1} B + \frac{1}{\varepsilon_k} E_1^T E_2)^T & 0 & -(R_{\varepsilon_k} + B^T Y_{k+1} B) \end{bmatrix} < 0, \quad (4.9)$$

where

$$R_{\varepsilon_k} = R + \frac{1}{\varepsilon_k} E_2^T E_2$$

$$Y_{k+1} = (S_{k+1}^{-1} - \varepsilon_k H H^T)^{-1} > 0.$$

Then, a control law given by

$$u_k = K_k x_k, \quad K_k = -(R_{\varepsilon_k} + B^T Y_{k+1} B)^{-1} \left( B^T Y_{k+1} A + \frac{1}{\varepsilon_k} E_2^T E_1 \right) \quad (4.10)$$

will achieve the following guaranteed cost:

$$J(N, d, \Delta A_k, \Delta B_k) \leq x_0^T S_0 x_0 \quad (4.11)$$

for any nonzero  $x_0$  and for all admissible uncertainties  $\Delta A_k$  and  $\Delta B_k$ .

Furthermore, for any quadratic guaranteed cost control  $u_k = \tilde{K}_k x_k$  of system (2.1) associated with cost matrix  $\tilde{S}_0$ , there exists some scaling sequence  $\varepsilon_k > 0$ ,  $k=0, 1, \dots, N-1$  such that  $S_0 < \tilde{S}_0$ ; that is

$$J(N, d, \Delta A_k, \Delta B_k) \leq x_0^T S_0 x_0 < x_0^T \tilde{S}_0 x_0$$

for any nonzero  $x_0$  and for all admissible uncertainties  $\Delta A_k$  and  $\Delta B_k$ .

**Proof.** By employing some algebraic manipulations, together with (4.10) and (4.8), we have from (4.9)

$$\begin{bmatrix} -S_k + \frac{1}{\varepsilon_k} (E_1 + E_2 K_k)^T (E_1 + E_2 K_k) + K_k^T R K_k + Q & (A + B K_k)^T \\ A + B K_k & -(S_{k+1}^{-1} - A_d A_d^T - \varepsilon_k H H^T) \end{bmatrix} < 0. \quad (4.12)$$

By the technique used in [5] and matrix inversion lemma, and taking into account (4.8), it follows from (4.12)

$$\begin{bmatrix} -S_k + K_k^T R K_k + Q & \hat{A}_k^T \\ A_k & -S_{k+1}^{-1} + A_d A_d^T \end{bmatrix} < 0, \quad (4.13)$$

where

$$\hat{A}_k = A + B K_k + H F_k (E_1 + E_2 K_k).$$

Next, note that the closed-loop system of (2.1) with control law (4.10) is given by

$$x_{k+1} = \hat{A}_k x_k + A_d x_{k-d}.$$

Then, by a similar proof as that of Theorem 4.1, it can be easily shown that

$$J(N, d, \Delta A_k, \Delta B_k) \leq x_0^T S_0 x_0, \quad x_0 \neq 0.$$

Finally, the rest part of the theorem can be worked out essentially following the same line as a result in the work of [39].  $\square$

**Remark 4.2.** Note that Theorem 4.3 provides a control design that gives a guaranteed cost for uncertain time-delay system (2.1) and establish the optimality of quadratic guaranteed cost control for the finite horizon case. Similarly to [39], the selection of the scaling sequence  $\varepsilon_k, k=0, 1, \dots, N-1$  is essential for the existence and optimization of a positive definite solution to (4.8)–(4.9). Indeed,

if  $x_0$  is a zero mean random variable with  $E(x_0 x_0^T) = I$ , where  $E(\cdot)$  denotes expectation, the cost function given by (4.11) can be rewritten as

$$J(N, d, \Delta A_k, \Delta B_k) \leq \text{tr}(S_0).$$

Furthermore, to minimize the cost  $\text{tr}(S_0)$ , we may modify an algorithm proposed by [39] to solve the following convex optimization problem:

$$\begin{aligned} \min_{\varepsilon_k} \quad & \text{tr}(S_k) \\ \text{s.t.} \quad & (4.8), (4.9). \end{aligned}$$

For more details, see [39].

Our last result deals with the problem of quadratic guaranteed cost control for infinite horizon case.

**Theorem 4.4.** Consider system (2.1) with cost function (2.5). Suppose for some scaling parameter  $\varepsilon > 0$ , there exists a positive definite solution  $Y$  to the algebraic Riccati inequalities

$$A_d^T Y A_d + \varepsilon H Y H^T - I < 0, \quad (4.14)$$

$$\begin{bmatrix} -Y + \frac{1}{\varepsilon} E_1^T E_1 + Q + A^T Y A & A^T Y B + \frac{1}{\varepsilon} E_1^T E_2 \\ (A^T Y B + \frac{1}{\varepsilon} E_1^T E_2)^T & -(R + \frac{1}{\varepsilon} E_2^T E_2 + B^T Y B) \end{bmatrix} < 0. \quad (4.15)$$

Then, a control law given by

$$u = Kx_k, \quad K = - \left( R + \frac{1}{\varepsilon} E_2^T E_2 + B^T Y B \right)^{-1} \left( B^T Y A + \frac{1}{\varepsilon} E_2^T E_1 \right) \quad (4.16)$$

will achieve the following guaranteed cost

$$J(\infty, d, \Delta A_k, \Delta B_k) \leq x_0^T S x_0, \quad S = (Y^{-1} + A_d^T A_d + \varepsilon H H^T)^{-1} \quad (4.17)$$

for any nonzero  $x_0$  and for all admissible uncertainties  $\Delta A_k$  and  $\Delta B_k$ .

Furthermore, for any quadratic guaranteed cost control  $u_k = \tilde{K}_k x_k$  of system (2.1) associated with cost matrix  $\tilde{S}$ , there exists some scaling parameter  $\varepsilon > 0$  such that  $S < \tilde{S}$ ; that is

$$J(\infty, d, \Delta A_k, \Delta B_k) \leq x_0^T S x_0 < x_0^T \tilde{S} x_0$$

for any nonzero  $x_0$  and for all admissible uncertainties  $\Delta A_k$  and  $\Delta B_k$ .

**Proof.** The desired result can be established along the same line as that in the proof of Theorem 4.3.  $\square$

**Remark 4.3.** It is shown in Theorems 4.3 and 4.4 that the quadratic guaranteed cost control problem for system (2.1) and the cost function (2.4) or (2.5) can be solved by the solution of some matrix inequalities. It should be pointed out that the guaranteed cost control (4.10) or (4.16) is delay  $d$  free (in general, the delay is not exactly known), so we may expect this control to be conservative

when the time-delay is known. Also, note that when  $A_d = 0$ , i.e., there is no time-delay in system (2.1), Theorems 4.3 and 4.4 will cover those in [39].

**Remark 4.4.** Similarly to Remark 4.2 as in the finite horizon case, to optimize the guaranteed cost of (4.17), we assume that  $x_0$  is a zero mean random variable with  $E(x_0 x_0^T) = I$ . Thus the bound in (4.17) will be replaced by  $\text{tr}(S)$ . Consequently, the optimal bound of (4.17) can be obtained by solving the following convex optimization problem [39]:

$$\begin{aligned} \min_{\varepsilon} \quad & \text{tr}(S) = \min_{\varepsilon} \text{tr}\{(Y^{-1} + A_d^T A_d + \varepsilon H H^T)^{-1}\} \\ \text{s.t.} \quad & (4.14), (4.15). \end{aligned}$$

**Remark 4.5.** It is worthwhile to mention that in traditional design for discrete time-delay systems, it was quite often to introduce an extra state, for example,  $z(t) = x(t-1)$ , such that the delay system can be transformed to an augmented nondelay system, then some existing techniques (e.g. optimal control, adaptive control, etc) can be applied to solve the corresponding stability and control problem. The drawback of the “augmented” method is: (1) the new “augmented” system would have a higher dimension, in particular, for a multiple time-delay system, which would increase the difficulty of design, consequently, the dimension of the controller, and (2) the “augmented” method cannot be applied to the situation when the time-delay factor is unknown, while the design method adopted in this paper can directly handle the time-delay term, and also take the unknown delay factor into account.

## 5. A numerical example

Consider the uncertain system (2.1) with cost function (2.5) with

$$A = \begin{bmatrix} -0.09 & 0.01 & 0.02 \\ 0.01 & -0.12 & 0.02 \\ 0.01 & 0.04 & -0.05 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.2 & 0 & 0 \\ 0 & -0.1 & 0.1 \\ 0 & 0 & -0.2 \end{bmatrix},$$

$$H = \begin{bmatrix} 0.1 \\ 0 \\ 0.2 \end{bmatrix}, \quad E_1 = [0.02 \ 0 \ 0.01], \quad E_2 = [2.5 \ 0.1 \ 1.6],$$

$$Q = \text{diag}[0.003, 0.003, 0.003], \quad R = \text{diag}[100, 100, 100], \quad \varepsilon = 0.01.$$

By solving inequalities (4.14)–(4.15), we obtain  $Y$  as

$$Y = \begin{bmatrix} 1.1249 & -0.0015 & 0.0347 \\ -0.0015 & 1.013 & -0.0023 \\ 0.0347 & -0.0023 & 2.0343 \end{bmatrix}.$$

Hence, we can calculate control gain  $K$  and the cost function  $S$  by (4.16) and (4.17):

$$K = \begin{bmatrix} -0.6342 & -0.0254 & -0.4062 \\ -0.0260 & -0.0007 & -0.0168 \\ -0.4071 & -0.0163 & -0.2602 \end{bmatrix}, \quad S = \begin{bmatrix} 0.9295 & 0.0013 & -0.0150 \\ 0.0013 & 0.9972 & -0.0089 \\ -0.0150 & -0.0089 & 0.5422 \end{bmatrix}.$$

Finally, we can work out the guaranteed cost,  $x_0^T S x_0$ , for the corresponding closed-loop system (2.1) with the controller from (4.16), for any given initial state  $x_0$ . For example, if the initial state  $x_0 = [1 \ 1 \ 1]^T$ , then the guaranteed cost function in (4.17) is

$$J(\infty, d, \Delta A_k, \Delta B_k) \leq x_0^T S x_0 = 2.4238.$$

## 6. Conclusion

In this paper, the problems of quadratically stability and guaranteed cost control for linear discrete uncertain time-delay systems are discussed. Matrix inequality conditions are proposed to ensure the uncertain system is quadratically stable regardless of all possible parameter uncertainties. Also, a control design method is presented such that the resulting closed-loop system has a upper bound of the cost function, which is in terms of solution of some matrix inequalities. In addition, the optimal guaranteed cost may be achieved by solving some convex optimization problem.

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